Explicit Liapunov Functions for a Class of Dissipative Systems with Circulatory Forces*

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Z. Naturforsch. **37a**, 196 (1982); received January 28, 1982

A class of dissipative systems with circulatory forces is given for which a necessary and sufficient condition for stability can be expressed as an explicit Liapunov function equivalent to the Hermitian form of a Hermitian matrix. For a discretized continuous system this matrix has a band structure.

In a previous note [1] on the stability of dissipative systems with circulatory forces a necessary and sufficient stability condition was derived. Such systems are of the form

$$\ddot{x} + (D + G)\,\dot{x} + (K + N)\,x = 0\,, \tag{1}$$

where x is an n-dimensional position vector and D, G, K, and N are time-independent $n \times n$ matrices with the properties that D is Hermitian positive definite, G is anti-Hermitian, K is Hermitian and N is anti-Hermitian. The statement proved in [1] is: if an anti-Hermitian operator T is a solution of

$$DT + TD + GT - TG = -2N \tag{2}$$

and if

$$(K + S)\,T - T\,(K + S) = 0\,, \tag{3}$$

then

$$(z, (K + T^2 + S)\,z) > 0 \tag{4}$$

is necessary and sufficient for the stability of the system (1) with (a, b) being the scalar product of a and b and

$$S = \tfrac{1}{2}(DT - TD) + \tfrac{1}{2}(GT + TG)\,. \tag{5}$$

The conditions (2) and (3) are generally not verified for arbitrarily given systems, just because of overdeterminacy. Even worse, it is not easy to test the conditions (2) and (3) in practice. What is more appealing is to solve this problem the other way around, which is the object of this note.

Suppose D, G and T are any matrices having the required properties, i.e. D is Hermitian positive-definite, G and T are anti-Hermitian. With the help of (2) we can easily compute N using just matrix products, as well as S using (5).

It remains to solve (3) for K. This is done by taking any analytic function or a polynomial over the eigenvalues of T for $K + S$, but retaining only the even powers of T in order to ensure symmetry for $K + S$, so that

$$K + S = f(T^2)\,. \tag{6}$$

Under certain conditions [2] $f(T^2)$ is the most general solution of (3).

Equation (6) allows us to construct general dissipative systems (sometimes the most general) with circulatory forces for which the stability conditions are fully known, i.e. (4). The left-hand side of (4) is a Hermitian form which makes the practical and numerical analysis of (1) much easier.

Discretization of Continuous Systems and their Stability

Systems with continuous operators having the properties of those of Eq. (1) happen in Fluid Dynamics and especially in plasma physics [3]. The practical stability analysis is done by discretizing the continuous differential operators and so replacing them by large band matrices.

If such a discretized system turns out to satisfy (6) with $f(T^2)$ taken as a polynomial and T a band matrix, then $K + S + T^2$ is a Hermitian band matrix. This would be great because very efficient numerical algorithms [4] have recently been found for the determination of the lowest eigenvalues of Hermitian band matrices.

Unfortunately, real discretized systems are not expected generally to satisfy Eqs. (3) or (6), and some complex eigenvalue codes will have to be applied to investigate their stability. The systems (6) found here explicitly could, however, help, even for very large matrices ($10^4 \times 10^4$), to test the less clear complex eigenvalue analysis.

* This work was performed under the terms of the agreement on association between Max-Planck-Institut für Plasmaphysik and EURATOM.

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